Twisted Partial Actions
A Classification of Stable C*-Algebraic Bundles

Peliminary Version

Ruy Exel

Departamento de Matemática
Universidade de São Paulo
C. P. 20570
01452-990 São Paulo Brazil
e-mail: exel@ime.usp.br

We introduce the notion of continuous twisted partial actions of a locally compact group on a C*-algebra. With such, we construct an associated C*-algebraic bundle called the semidirect product bundle. Our main theorem shows that, given any C*-algebraic bundle which is second countable and whose unit fiber algebra is stable, there is a continuous twisted partial action of the base group on the unit fiber algebra, whose associated semidirect product bundle is isomorphic to the given one.

1. Introduction

A C^*-algebraic bundle is, roughly speaking, a natural generalization of the concept of graded C^*-algebras, to the case when the grading group is a locally compact group. A C^*-algebraic bundle $\mathcal{B}$ over the group G consists, therefore, of a collection of Banach spaces $(B_t)_{t\in G}$ which are glued together to form a Banach bundle [**4**, II.13.4] and which moreover comes equipped with a family of multiplication operations

$$\cdot : B_r \times B_s \to B_{rs}, \quad r, s \in G$$

and a family of involution operations

$$* : B_t \to B_{t^{-1}}, \quad t \in G$$

all of which are continuous with respect to r, s and t, satisfying axioms that are modeled after the properties which would be satisfied, were the B_t's the grading subspaces of a graded *-algebra $\bigoplus_{t\in G} B_t$.

C^*-algebraic bundles naturally occur in a large number of situations. First of all, they show up in connection with the theory of group representations, as carefully described in

the comprehensive two volume book by Fell and Doran, listed below as reference number [**4**]. In the discrete group case, there is a very close relationship (although not a perfect equivalence [**4**, VIII.16.12]) between C^*-algebraic bundles and graded C^*-algebras. The latter, in turn, appears in connection to the theory of group actions on C^*-algebras. In fact, whenever a compact abelian group K acts on a C^*-algebra B, then there is a natural grading on B by the dual group $G = \hat{K}$, given by the spectral subspaces (see, for example [**3**, Section 2] for the case of the circle group). More generally, Quigg [**8**] has shown that a co-action of a discrete group (a concept which generalizes actions of the not always visible compact dual group) also yields graded C^*-algebras.

In this work we propose to extend our earlier work [**3**] on circle actions (i.e. $\mathbf{Z}$-graded algebras) to the context of general C^*-algebraic bundles. As a result, we obtain a classification theorem which exhibits any C^*-algebraic bundle, satisfying certain mild hypothesis, as the *semidirect product bundle* for a continuous *twisted partial action* of the base group on the unit fiber algebra. The concept of twisted partial actions is a simultaneous generalization of the twisted crossed-products of Zeller-Meier [**12**] on one hand and the partial actions which we introduced in [**3**] on the other (see also the work of Packer and Raeburn [**6**] for the twisted case, and McClanahan's work [**5**] on partial actions of discrete groups).

A continuous twisted partial action of a locally compact group G on a C^*-algebra A consists of a family $\{D_t\}_{t\in G}$ of closed two sided ideals of A, a family $\{\theta_t\}_{t\in G}$ of isomorphisms from $D_{t^{-1}}$ to D_t and a "cocycle" $w = \{w(r,s)\}_{(r,s)\in G\times G}$, where each $w(r,s)$ is a unitary multiplier of the ideal $D_r \cap D_{rs}$, which satisfy properties similar to the axioms of twisted actions (see below for more details).

Given such an object, we construct a C^*-algebraic bundle called the semidirect product bundle of A and G, generalizing [**4**, VIII.4]. The power of this construction is such that we are able, in turn, to show, in our main Theorem, that every second countable C^*-algebraic bundle, whose unit fiber algebra is stable, can be obtained as the result of our construction.

The requirement that the unit fiber algebra be compact can obviously be dropped if one is willing to "stabilize" the given bundle, by tensoring it with the algebra of compact operators on a separable Hilbert space. Given this, our theorem can then be applied to virtually all C^*-algebraic bundles.

As far as introducing a generalized partial crossed-product algebra (which we did in [**3**] in the case of partial actions of the integers or was done in [**5**] for partial actions by general discrete groups), observe that the usual process of forming the crossed-product of a C^*-algebra by a group action [**7**] can be divided in two steps, the first one being the construction of the associated semidirect product bundle as in [**4**, VIII.4]. The second step is then to form the cross-sectional algebra [**4**, VII.5], which is a process that can be applied to any C^*-algebraic bundle, irrespective of how it came about.

That is, we may stop short of defining the concept of crossed-products by twisted partial actions, since the associated semidirect product bundle can then be fed to the machinery of cross-sectional algebras, which would, via a standard procedure, produce what we would call the crossed-product C^*-algebra of a locally compact group by a continuous

twisted partial action. In particular, we completely avoid, in this way, the usual problems caused by non-amenable groups.

2. The Discrete Group case

The axioms for continuous twisted partial actions of locally compact groups and the basic work leading to the construction of the associated C^*-algebraic bundle can be divided in two distinct parts, the first one relating to algebraic properties and the second referring to topological aspects. In order to organize the exposition, we have, therefore, chosen to break up the presentation of the definition and basic properties of this concept in two sections, the present one being dedicated to the algebraic considerations. We thus restrict our initial discussion to groups without topology or, what amounts to the same, to discrete groups.

Let A be a C^*-algebra and let G be a discrete group.

2.1. Definition. *A twisted partial action of G on A is a triple*

$$\Theta = \big(\{D_t\}_{t\in G}, \{\theta_t\}_{t\in G}, \{w(r,s)\}_{(r,s)\in G\times G}\big)$$

*where for each t in G, D_t is a closed two sided ideal in A, θ_t is a *-isomorphism from $D_{t^{-1}}$ onto D_t and for each (r,s) in $G\times G$, $w(r,s)$ is a unitary multiplier of $D_r\cap D_{rs}$, satisfying the following postulates, for all r, s and t in G*

- *a)* $D_e = A$ *and* θ_e *is the identity automorphism of* A *.*
- *b)* $\theta_r(D_{r^{-1}}\cap D_s) = D_r\cap D_{rs}$
- *c)* $\theta_r(\theta_s(a)) = w(r,s)\theta_{rs}(a)w(r,s)^*, \quad a\in D_{s^{-1}}\cap D_{s^{-1}r^{-1}}$
- *d)* $w(e,t) = w(t,e) = 1$
- *e)* $\theta_r(aw(s,t))w(r,st) = \theta_r(a)w(r,s)w(rs,t), \quad a\in D_{r^{-1}}\cap D_s\cap D_{st}$

Our goal is to construct, given a twisted partial action of G on A, a C^*-algebraic bundle over G. We recall below, the definition of this concept in the special case of discrete groups [**4**, II.13.1, II.13.4, VIII.2.2, VIII.3.1, VIII.16.2].

2.2. Definition. *A C^*-algebraic bundle over a discrete group G is a collection of Banach spaces $\{B_t\}_{t\in G}$ together with a multiplication operation*

$$\cdot : \mathcal{B}\times\mathcal{B}\to\mathcal{B}$$

and an involution

$$* : \mathcal{B}\to\mathcal{B}$$

where $\mathcal{B}$ is the disjoint union of all B_t's, satisfying for all r, s and t in G and b and c in $\mathcal{B}$

- *i)* $B_rB_s\subseteq B_{rs}$

ii) The product $\cdot$ *is bilinear on* $B_r \times B_s$ *to* B_{rs} .

iii) The product on $\mathcal{B}$ *is associative.*

iv) $\|bc\| \leq \|b\|\|c\|$

v) $(B_t)^* \subseteq B_{t^{-1}}$

vi) $*$ *is conjugate-linear from* B_t *to* $B_{t^{-1}}$

vii) $(bc)^* = c^*b^*$

viii) $b^{**} = b$

ix) $\|b^*\| = \|b\|$

x) $\|bb^*\| = \|b\|^2$

xi) $bb^* \geq 0$ *in* B_e

Axioms (i)–(iv) define what is called a Banach algebraic bundle [**4**, VIII.2.2]. Adding (v)–(ix) gives the definition of a Banach *-algebraic bundle [**4**, VIII.3.1] while the last two properties characterize C^*-algebraic bundles [**4**, VIII.16.2]. In the above definition we have omitted all references to continuity, since we are, for the time being, considering exclusively discrete groups. See section (3) below for the general case.

We shall denote both the family $\{B_t\}_{t\in G}$ and the disjoint union of the $B_t's$ by $\mathcal{B}$ as this will not bring any confusion.

Fix, from now on, a C^*-algebra A, a discrete group G and a twisted partial action of G on A given by
$$\Theta = \big(\{D_t\}_{t\in G}, \{\theta_t\}_{t\in G}, \{w(r,s)\}_{(r,s)\in G\times G}\big) .$$
As a first step in constructing a C^*-algebraic bundle from Θ, let
$$\mathcal{B} = \{(a,s) \in A \times G : a \in D_s\},$$
and for each t in G, let B_t be the subset of $\mathcal{B}$ formed by all the (a,s) with $s = t$. We will also use the notation $a\delta_t$ for (a,t), whence $B_t = D_t\delta_t$.

There is an obvious bijection between B_t and D_t, through which we give B_t the structure of a Banach space.

Let us define the multiplication operation on $\mathcal{B}$ by
$$(a_r\delta_r) * (b_s\delta_s) = \theta_r\left(\theta_r^{-1}(a_r)b_s\right) w(r,s)\delta_{rs}$$
for a_r in D_r and b_s in D_s. It is important to remark that the term $\theta_r\left(\theta_r^{-1}(a_r)b_s\right)$ belongs to $D_r \cap D_{rs}$ by (2.1.b) and hence that multiplication of this term by $w(r,s)$ is well defined. The result lying again in $D_r \cap D_{rs}$, guarantees that the right hand side above in fact gives an element in B_{rs}. We have thus verified (2.2.i).

The involution on $\mathcal{B}$ is defined by
$$(a_t\delta_t)^* = \theta_t^{-1}(a_t^*)w(t^{-1},t)^*\delta_{t^{-1}}$$

for a_t in D_t. It should be noted that this gives $(B_t)^* \subseteq B_{t^{-1}}$, hence proving (2.2.v).

Also observe that (2.1.c) with $r = t^{-1}$, $s = t$ and $a = \theta_t^{-1}(a_t^*)$ provides

$$\theta_{t^{-1}}(a_t^*) = w(t^{-1}, t)\theta_t^{-1}(a_t^*)w(t^{-1}, t)^*$$

and so the definition above is equivalent to

$$(a_t\delta_t)^* = w(t^{-1}, t)^*\theta_{t^{-1}}(a_t^*)\delta_{t^{-1}}.$$

One should be careful not to mistake $\theta_{t^{-1}}$ for θ_t^{-1} which, as seen above, coincide only up to conjugation by $w(t^{-1}, t)$.

This specifies all of the required ingredients of a C^*-algebraic bundle and we must therefore verify the validity of properties (i)–(xi) above. Apart from (i) and (v) which have already been checked, note that (ii), (iv), (vi) and (ix) can all be proved without much effort. In contrast, proving associativity and the anti-multiplicativity of the involution is a bit of a challenge.

2.3. Lemma. *If $\{u_i\}_i$ is an approximate identity (always assumed to be self-adjoint and of norm one) for $D_{r^{-1}}$ and if a_r and b_s are elements of D_r and D_s, respectively, then*

$$(a_r\delta_r) * (b_s\delta_s) = \lim_i a_r\theta_r(u_ib_s)w(r, s)\delta_{rs}.$$

Proof. We have

$$(a_r\delta_r) * (b_s\delta_s) = \theta_r\left(\theta_r^{-1}(a_r)b_s\right)w(r, s)\delta_{rs}$$

$$= \lim_i \theta_r\left(\theta_r^{-1}(a_r)u_ib_s\right)w(r, s)\delta_{rs} = \lim_i a_r\theta_r(u_ib_s)w(r, s)\delta_{rs}. \qquad \square$$

2.4. Proposition. *The multiplication defined above is associative.*

Proof. Let a_r, b_s and c_t be in D_r, D_s and D_t, respectively. Also let $\{u_i\}_i$ be an approximate identity for $D_{r^{-1}}$. Then we have

$$(a_r\delta_r * b_s\delta_s) * c_t\delta_t = \left(\lim_i a_r\theta_r(u_ib_s)w(r, s)\delta_{rs}\right) * c_t\delta_t = \ldots$$

Let $x_i = a_r\theta_r(u_ib_s)w(r, s)$. So the above equals

$$\ldots = \lim_i x_i\delta_{rs} * c_t\delta_t = \lim_i \theta_{rs}\left(\theta_{rs}^{-1}(x_i)c_t\right)w(rs, t)\delta_{rst} = \ldots$$

Note that x_i is in $D_r \cap D_{rs}$ so that $\theta_{rs}^{-1}(x_i)$ is in $D_{s^{-1}} \cap D_{s^{-1}r^{-1}}$. Let v_j be an approximate identity for $D_{s^{-1}} \cap D_{s^{-1}r^{-1}}$. So the above equals

$$\ldots = \lim_i \lim_j \theta_{rs}\left(\theta_{rs}^{-1}(x_i)v_jc_t\right)w(rs, t)\delta_{rst}$$

$$= \lim_i \lim_j a_r\theta_r(u_ib_s)w(r, s)\theta_{rs}(v_jc_t)w(rs, t)\delta_{rst}$$

$$= \lim_i \lim_j a_r \theta_r(u_i b_s)\theta_r\left(\theta_s(v_j c_t)\right) w(r,s)w(rs,t)\delta_{rst}$$

$$= \lim_i \lim_j a_r \theta_r\left[u_i b_s \theta_s(v_j c_t)\right] w(r,s)w(rs,t)\delta_{rst}$$

$$= \lim_i \lim_j a_r \theta_r\left[\theta_s\left(\theta_s^{-1}(u_i b_s) v_j c_t\right)\right] w(r,s)w(rs,t)\delta_{rst} = \ldots$$

Note that $u_i b_s$ is in $D_{r^{-1}} \cap D_s$ so that $\theta_s^{-1}(u_i b_s)$ is in $D_{s^{-1}} \cap D_{s^{-1}r^{-1}}$ so the above equals

$$\ldots = \lim_i a_r \theta_r\left[\theta_s\left(\theta_s^{-1}(u_i b_s) c_t\right)\right] w(r,s)w(rs,t)\delta_{rst} = \ldots$$

Without loss of generality we may assume that $b_s = b_s' b_s''$ where both b_s' and b_s'' belong to D_s. So the above equals

$$\ldots = \lim_i \theta_r\left[\theta_r^{-1}(a_r)\theta_s\left(\theta_s^{-1}(u_i b_s')\theta_s^{-1}(b_s'')c_t\right)\right] w(r,s)w(rs,t)\delta_{rst}$$

$$= \lim_i \theta_r\left[\theta_r^{-1}(a_r) u_i b_s' \theta_s\left(\theta_s^{-1}(b_s'')c_t\right)\right] w(r,s)w(rs,t)\delta_{rst}$$

$$= \theta_r\left[\theta_r^{-1}(a_r) b_s' \theta_s\left(\theta_s^{-1}(b_s'')c_t\right)\right] w(r,s)w(rs,t)\delta_{rst}$$

$$= \theta_r\left[\theta_r^{-1}(a_r) \theta_s\left(\theta_s^{-1}(b_s)c_t\right)\right] w(r,s)w(rs,t)\delta_{rst}.$$

On the other hand

$$a_r\delta_r * (b_s\delta_s * c_t\delta_t) = a_r\delta_r * \theta_s\left(\theta_s^{-1}(b_s)c_t\right) w(s,t)\delta_{st}$$

$$= \theta_r\left[\theta_r^{-1}(a_r)\theta_s\left(\theta_s^{-1}(b_s)c_t\right) w(s,t)\right] w(r,st)\delta_{rst} = \ldots$$

If $x = \theta_r^{-1}(a_r)\theta_s\left(\theta_s^{-1}(b_s)c_t\right)$, then x is in $D_{r^{-1}} \cap D_s \cap D_{st}$ so by (2.1.e) the above equals

$$\ldots = \theta_r(x) w(r,s)w(rs,t)\delta_{rst} = \theta_r\left[\theta_r^{-1}(a_r)\theta_s\left(\theta_s^{-1}(b_s)c_t\right)\right] w(r,s)w(rs,t)\delta_{rst}. \qquad \square$$

Let us now state an identity to be used in the proof of the anti-multiplicativity of the involution. The proof is omitted as this is basically a rewriting of (2.1.e).

2.5. Lemma. *If f, g and h are in G, then*

$$\theta_f\left(a w(g,h)^*\right) = \theta_f(a) w(f,gh) w(fg,h)^* w(f,g)^*, \quad a \in D_{f^{-1}} \cap D_g \cap D_{gh}.$$

2.6. Proposition. *The involution defined above is anti-multiplicative.*

Proof. Let a_r and b_s be in D_r and D_s, respectively. We want to prove that
$$(a_r\delta_r * b_s\delta_s)^* = (b_s\delta_s)^* * (a_r\delta_r)^*.$$
The left hand side equals
$$\left[\theta_r\left(\theta_r^{-1}(a_r)b_s\right)w(r,s)\delta_{rs}\right]^* = \theta_{rs}^{-1}\left[w(r,s)^*\theta_r\left(b_s^*\theta_r^{-1}(a_r^*)\right)\right]w(s^{-1}r^{-1},rs)^*\delta_{s^{-1}r^{-1}}.$$
Let $x = b_s^*\theta_r^{-1}(a_r^*)$. Then x is in $D_s \cap D_{r^{-1}}$ whence $\theta_s^{-1}(x)$ is in $D_{s^{-1}} \cap D_{s^{-1}r^{-1}}$ and we have by axiom (2.1.c)
$$\theta_r(x) = \theta_r\left(\theta_s\left(\theta_s^{-1}(x)\right)\right) = w(r,s)\theta_{rs}\left(\theta_s^{-1}(x)\right)w(r,s)^*$$
and thus
$$\begin{aligned}(a_r\delta_r * b_s\delta_s)^* &= \theta_{rs}^{-1}\left[\theta_{rs}\left(\theta_s^{-1}(x)\right)w(r,s)^*\right]w(s^{-1}r^{-1},rs)^*\delta_{s^{-1}r^{-1}}\\ &= w(s^{-1}r^{-1},rs)^*\theta_{s^{-1}r^{-1}}\left[\theta_{rs}\left(\theta_s^{-1}(x)\right)w(r,s)^*\right]\delta_{s^{-1}r^{-1}} = \dots\end{aligned}$$
Using (2.5) with $f = s^{-1}r^{-1}$, $g = r$, $h = s$ and $a = \theta_{rs}\left(\theta_s^{-1}(x)\right)$ the above equals
$$\begin{aligned}\dots &= w(s^{-1}r^{-1},rs)^*\theta_{s^{-1}r^{-1}}\left[\theta_{rs}\left(\theta_s^{-1}(x)\right)\right]w(s^{-1}r^{-1},rs)w(s^{-1},s)^*w(s^{-1}r^{-1},r)^*\delta_{s^{-1}r^{-1}}\\ &= \theta_{rs}^{-1}\left[\theta_{rs}\left(\theta_s^{-1}(x)\right)\right]w(s^{-1},s)^*w(s^{-1}r^{-1},r)^*\delta_{s^{-1}r^{-1}}\\ &= \theta_s^{-1}(x)w(s^{-1},s)^*w(s^{-1}r^{-1},r)^*\delta_{s^{-1}r^{-1}}\\ &= \theta_s^{-1}\left(b_s^*\theta_r^{-1}(a_r^*)\right)w(s^{-1},s)^*w(s^{-1}r^{-1},r)^*\delta_{s^{-1}r^{-1}}.\end{aligned}$$
On the other hand
$$\begin{aligned}(b_s\delta_s)^* * (a_r\delta_r)^* &= \left(\theta_s^{-1}(b_s^*)w(s^{-1},s)^*\delta_{s^{-1}}\right) * \left(\theta_r^{-1}(a_r^*)w(r^{-1},r)^*\delta_{r^{-1}}\right)\\ &= \left(w(s^{-1},s)^*\theta_{s^{-1}}(b_s^*)\delta_{s^{-1}}\right) * \left(\theta_r^{-1}(a_r^*)w(r^{-1},r)^*\delta_{r^{-1}}\right)\\ &= \theta_{s^{-1}}\left\{\theta_{s^{-1}}^{-1}\left[w(s^{-1},s)^*\theta_{s^{-1}}(b_s^*)\right]\theta_r^{-1}(a_r^*)w(r^{-1},r)^*\right\}w(s^{-1}r^{-1})\delta_{s^{-1},r^{-1}} = \dots\end{aligned}$$
Let $x = \theta_{s^{-1}}(b_s^*)$ and $y = \theta_r^{-1}(a_r^*)w(r^{-1},r)^*$ and let u_i be an approximate identity for D_s. So the above equals
$$\begin{aligned}\dots &= \lim_i \theta_{s^{-1}}\left\{\theta_{s^{-1}}^{-1}\left[w(s^{-1},s)^*x\right]u_iy\right\}w(s^{-1},r^{-1})\delta_{s^{-1}r^{-1}}\\ &= \lim_i \theta_{s^{-1}}\left\{\theta_{s^{-1}}^{-1}\left[w(s^{-1},s)^*x\theta_{s^{-1}}(u_iy)\right]\right\}w(s^{-1},r^{-1})\delta_{s^{-1}r^{-1}}\\ &= \lim_i w(s^{-1},s)^*x\theta_{s^{-1}}(u_iy)w(s^{-1},r^{-1})\delta_{s^{-1}r^{-1}}\\ &= \lim_i w(s^{-1},s)^*\theta_{s^{-1}}(b_s^*u_iy)w(s^{-1},r^{-1})\delta_{s^{-1}r^{-1}}\\ &= w(s^{-1},s)^*\theta_{s^{-1}}(b_s^*y)w(s^{-1},r^{-1})\delta_{s^{-1}r^{-1}}\\ &= w(s^{-1},s)^*\theta_{s^{-1}}\left[b_s^*\theta_r^{-1}(a_r^*)w(r^{-1},r)^*\right]w(s^{-1},r^{-1})\delta_{s^{-1}r^{-1}} = \dots\end{aligned}$$
Let us now use lemma (2.5) once more with $f = s^{-1}$, $g = r^{-1}$, $h = r$ and $a = b_s^*\theta_r^{-1}(a_r^*)$ to conclude that the above equals
$$\begin{aligned}\dots &= w(s^{-1},s)^*\theta_{s^{-1}}\left(b_s^*\theta_r^{-1}(a_r^*)\right)w(s^{-1},e)w(s^{-1}r^{-1},r)^*w(s^{-1},r^{-1})^*w(s^{-1},r^{-1})\delta_{s^{-1}r^{-1}}\\ &= \theta_s^{-1}\left(b_s^*\theta_r^{-1}(a_r^*)\right)w(s^{-1},s)w(s^{-1}r^{-1},r)^*\delta_{s^{-1}r^{-1}}.\end{aligned}$$
□

We leave for the reader to verify the remaining properties, i.e., (viii), (x) and (ix) of Definition (2.2). Once this is done we have proven the main result of this section:

2.7. Theorem. *Given a twisted partial action*

$$\Theta = \big(\{D_t\}_{t\in G}, \{\theta_t\}_{t\in G}, \{w(r,s)\}_{(r,s)\in G\times G}\big)$$

of the discrete group G on A, the bundle $\mathcal{B} = \{D_t\delta_t\}_{t\in G}$ is a C^*-algebraic bundle over G with the operations

$$(a_r\delta_r) * (b_s\delta_s) = \theta_r\left(\theta_r^{-1}(a_r)b_s\right) w(r,s)\delta_{rs}$$

and

$$(a_t\delta_t)^* = \theta_t^{-1}(a_t^*)w(t^{-1},t)^*\delta_{t^{-1}}.$$

2.8. Definition. *The C^*-algebraic bundle constructed above will be called the semidirect product bundle of A and G (after* [**4**, VIII.4]*).*

This concludes the algebraic part of our construction.

3. The Continuous Group case

From now on we will let G be a locally compact topological group. Of course one would like to add extra requirements to the definition of twisted partial actions to account for the topology of G. In other words we would like to require that twisted partial actions be continuous in a suitable sense to be made precise below.

To begin with, let us establish the relevant concept of continuity for a family of subspaces of a given Banach space. Let, therefore, E be a Banach space and let $\{E_x\}_{x\in X}$ be a family of linear subspaces of E, indexed by a topological space X.

3.1. Definition. *We say that $\{E_x\}_{x\in X}$ is continuous if, for any open set $U \subseteq E$, the set*

$$\{x \in X : E_x \cap U \neq \emptyset\}$$

is open in X.

Consider the subset $\mathcal{E}$ of $E \times X$ defined by

$$\mathcal{E} = \{(v,x) \in E \times X : v \in E_x\}$$

equipped with the relative topology from $E\times X$. If continuity is assumed, we claim that $\mathcal{E}$ is a Banach bundle over X, as defined in [**4**, II.13.4]. In fact, the properties (i)–(iv) in that definition are automatically verified as they all follow from the corresponding facts which hold for the trivial Banach bundle $E \times X$. The main question hinges on the openness of the bundle projection.

3.2. Proposition. *The family $\{E_x\}_{x\in X}$ is continuous if and only if the bundle projection*

$$\pi\colon (v,x) \in \mathcal{E} \to x \in X$$

is an open map. In this case $\mathcal{E}$ is a Banach bundle over X.

Proof. The proof is elementary and hence omitted. □

Observe that a section $\gamma: X \to E$ must necessarily have the form $\gamma(x) = (\beta(x), x)$ for some function $\beta: X \to E$ such that $\beta(x) \in E_x$ for all x in X. Since $\mathcal{E}$ has the relative topology, one sees that γ is continuous if and only if β is. Given the very close relationship between γ and β, they will be deliberately confused with each other.

3.3. Proposition. *If for any x_0 in X, any $\varepsilon > 0$ and any v in E_{x_0} there exists a continuous section β such that $\|\beta(x_0) - v\| < \varepsilon$, then $\{E_x\}_{x\in X}$ is continuous. In addition, if X is either locally compact or paracompact, then the converse holds even strongly in the sense that a section β can always be found with $\beta(x_0) = v$.*

Proof. Let U be an open subset of E and suppose that x_0 is such that $E_{x_0} \cap U \neq \emptyset$. Pick a continuous section β such that $\beta(x_0) \in E_{x_0} \cap U$. Note that for any x in $\beta^{-1}(U)$ one has that $\beta(x) \in E_x \cap U$ and consequently $E_x \cap U$ is non-empty. If we note that $\beta^{-1}(U)$ is open, we see that the the first part of the statement is proven. Conversely, if $\{E_x\}_{x\in X}$ is continuous, then by (3.2), $\mathcal{E}$ is a Banach bundle over X. The conclusion, then follows from a result of Douady and Dal Soglio-Hérault, stating that Banach bundles over locally compact or paracompact base spaces have plenty of continuous sections [**4**, II.13.19]. □

In connection to this let us define, for future reference, the concept of pointwise-dense set of sections.

3.4. Definition. *A set Γ of continuous sections of a given Banach bundle is said to be pointwise-dense if for any x in the base space, the set $\{\gamma(x): \gamma \in \Gamma\}$ is dense in the fiber over x.*

Returning to the case of our twisted partial action, let us assume henceforth that the collection of ideals $\{D_t\}_{t\in G}$ is continuous in the sense above. Since the group is assumed to be locally compact we will thus have continuous sections in abundance.

As in the previous section, let $\mathcal{B} = \{(a,t) \in A \times G : a \in D_t\}$ which we now consider as a topological subspace of $A \times G$. It then follows from (3.2) that $\mathcal{B}$ is a Banach bundle over G.

Because the inversion map is continuous on G, the same reasoning above shows that $\mathcal{B}^{-1} = \{(a,t) \in A \times G : a \in D_{t^{-1}}\}$ is also a Banach bundle, and then the family of isomorphisms $\{\theta_t\}_{t\in G}$ can be used to define a bundle map $\theta: \mathcal{B}^{-1} \to \mathcal{B}$.

3.5. Definition. *We say that $\{\theta_t\}_{t\in G}$ is continuous if the corresponding map*

$$\theta: (a,t) \in \mathcal{B}^{-1} \to (\theta_t(a), t) \in \mathcal{B}$$

is continuous.

Note that [**4**, II.13.16] provides the following equivalent characterization of continuity:

3.6. Proposition. *Suppose that Γ is a fixed pointwise-dense space of sections for $\mathcal{D}^{-1}$. Then $\{\theta_t\}_{t\in G}$ is continuous if and only if for any γ in Γ one has that the map $t \in G \mapsto \theta_t(\gamma(t)) \in A$ is continuous.*

From [**4**, II.13.17] it follows that, if θ is continuous, then its inverse is continuous as well.

We must now discuss continuity for $\{w(r,s)\}_{(r,s)\in G\times G}$. The idea again will be to define continuity in terms of the continuity of the corresponding bundle map. However, there is a slight problem here because $w(r,s)$ is a map (actually a multiplier consists of a pair of maps) defined in $D_r \cap D_{rs}$ and one should worry in the first place whether or not these form a Banach bundle. The question here is whether the pointwise intersection of two continuously varying families of subspaces is again continuous in our sense. In general this is not the case but, fortunately, this is true for ideals. In fact, using (3.3) and recalling that the intersection of two ideals equals their product, one obtains enough sections of the intersection bundle by multiplying together a pair of sections of each bundle. So, let $\mathcal{D}$ be the Banach bundle over $G\times G$ having $D_r \cap D_{rs}$ as the fiber over (r,s) in the spirit of (3.2). The family $\{w(r,s)\}_{(r,s)\in G\times G}$ then defines bundle maps

$$L, R\colon \mathcal{D} \to \mathcal{D}$$

given by the left and right action of the multipliers $w(r,s)$, respectively.

3.7. Definition. *We say that $\{w(r,s)\}_{(r,s)\in G\times G}$ is continuous if both L and R are continuous maps from $\mathcal{D}$ to $\mathcal{D}$. Equivalently (see* [**4**, II.13.16]*), if for any γ in a fixed pointwise-dense set of sections of $\mathcal{D}$ one has that both*

$$(r,s) \in G \mapsto \gamma(r,s)w(r,s) \in A$$

and

$$(r,s) \in G \mapsto w(r,s)\gamma(r,s) \in A$$

are continuous.

The definition of continuity for twisted partial actions is thus obtained by requiring that all of its components be continuous in the appropriate senses:

3.8. Definition. *If $\Theta = \big(\{D_t\}_{t\in G}, \{\theta_t\}_{t\in G}, \{w(r,s)\}_{(r,s)\in G\times G}\big)$ is a twisted partial action of the locally compact group G on the C^*-algebra A, we say that Θ is continuous if*

- *a)* $\{D_t\}_{t\in G}$ *is continuous in the sense of (3.1).*
- *b)* $\{\theta_t\}_{t\in G}$ *is continuous in the sense of (3.5).*
- *c)* $\{w(r,s)\}_{(r,s)\in G\times G}$ *is continuous in the sense of (3.7).*

In the special case in which the action is neither twisted nor partial, that is, if $w(r,s) = 1$ and $D_r = A$ for all r and s in G, observe that our definition of continuity reduces to the usual concept of strongly continuous group action.

Let us now fix a continuous twisted partial action Θ of the locally compact group G on A. Our goal will be to show that the semidirect product bundle of section (2), equipped with the relative topology (from $G \times A$) is a continuous C^*-algebraic bundle over G. The above considerations, in particular Proposition (3.2), already tells us that $\mathcal{B}$ is a Banach bundle over G. For the convenience of the reader let us recall the definition of a continuous C^*-algebraic bundle.

3.9. Definition. *A continuous C^*-algebraic bundle over the locally compact group G is a Banach bundle $\mathcal{B} = \{B_t\}_{t\in G}$ together with a continuous multiplication operation*

$$\cdot : \mathcal{B} \times \mathcal{B} \to \mathcal{B}$$

and a continuous involution

$$* : \mathcal{B} \to \mathcal{B}$$

satisfying (i)–(xi) of (2.2).

The following concludes the presentation of our main construction.

3.10. Theorem. *Let $\Theta = \big(\{D_t\}_{t\in G}, \{\theta_t\}_{t\in G}, \{w(r,s)\}_{(r,s)\in G\times G}\big)$ be a continuous twisted partial action of the locally compact group G on the C^*-algebra A. Then the semidirect product bundle of A and G, with the relative topology of $A\times G$, is a continuous C^*-algebraic bundle over G.*

Proof. After the work done in section (2), it is enough to verify the continuity of the multiplication and of the involution. In order to do this, we use [**4**, VIII.2.4 and 3.2]. Assume that α and β are continuous sections of $\mathcal{B}$. So $\alpha(t) = a_t\delta_t$ and $\beta(t) = b_t\delta_t$ where a_t and b_t are continuous A-valued functions on G with $a_t, b_t \in D_t$ for all t. We must therefore prove that the map $(r,s) \in G \mapsto \alpha(r)\beta(s) \in \mathcal{B}$ is continuous. By definition, we have $\alpha(r)\beta(s) = \theta_r\left(\theta_r^{-1}(a_r)b_s\right) w(r,s)\delta_{rs}$. Now, note that the map

$$\gamma\colon (r,s) \in G \times G \mapsto \theta_r\left(\theta_r^{-1}(a_r)b_s\right) \in A$$

is continuous. Also, observe that, since $\gamma(r,s) \in D_r \cap D_{rs}$, we see that γ gives a continuous section of the bundle $\mathcal{D}$, mentioned before (3.7). The continuity of $w(r,s)$ can now be invoked to conclude that $\alpha(r)\beta(s)$ is continuous.

With respect to the involution, using [**4**, VIII.3.2], we must show that for each continuous section $\alpha(t) = a_t\delta_t$, one has that $\alpha(t)^*$ is continuous. Recall that $\alpha(t)^* = \theta_t^{-1}(a_t^*)w(t^{-1},t)^*\delta_{t^{-1}}$. So proving continuity of $\alpha(t)^*$ amounts to proving the continuity of the A-valued function

$$t \in G \mapsto \theta_t^{-1}(a_t^*)w(t^{-1},t)^* \in A.$$

Note that this map is given by the composition of the continuous map

$$t \in G \mapsto \left((t^{-1},t), \theta_t^{-1}(a_t^*)\right) \in \mathcal{D}$$

followed by the inverse of the map L mentioned in (3.7), and finally, the projection from $\mathcal{D}$ to A. $\square$

4. Ternary Rings of Operators

If $\mathcal{B}$ is a C^*-algebraic bundle then, except for the unit fiber algebra B_e, the fibers B_t are not closed under multiplication and, therefore, do not possess the structure of an algebra. Nevertheless B_t has a rich algebraic structure provided by the ternary operation xy^*z, for $x, y, z \in B_t$, with respect to which it is closed. This makes B_t a ternary C^*-ring as defined by Zettl [**13**].

4.1. Definition. *A ternary C^*-ring is a complex Banach space E, equipped with a ternary operation*

$$(a, b, c) \in E \times E \times E \mapsto a \cdot b \cdot c \in E$$

which is linear in the first and third variables, conjugate linear in the middle variable and which satisfies the following for all $a, b, c, d, e \in E$

i) $(a \cdot b \cdot c) \cdot d \cdot e = a \cdot (d \cdot c \cdot b) \cdot e = a \cdot b \cdot (c \cdot d \cdot e)$

ii) $\|a \cdot b \cdot c\| \leq \|a\| \, \|b\| \, \|c\|$

iii) $\|a \cdot a \cdot a\| = \|a\|^3$

According to Theorem (3.1) in [**13**], there is a fundamental dichotomy in the theory of ternary C^*-rings in the sense that any such object is the direct sum of a *ternary ring of operators* (TRO) and what could be called an anti-TRO. The definitions are as follows:

4.2. Definition. *A ternary ring of operators is a closed subspace E of operators on a Hilbert space H (Zettl considers the case of operators between two Hilbert spaces but this is not a crucial matter) such that $EE^*E \subseteq E$, equipped with the ternary operation*

$$a \cdot b \cdot c = ab^*c, \quad a, b, c \in E.$$

An anti-TRO is a TRO except that the ternary operation, with which it comes equipped, is

$$a \cdot b \cdot c = -ab^*c, \quad a, b, c \in E.$$

Clearly TROs as well as anti-TROs are examples of ternary C^*-rings. It is interesting to remark, however, that there is a legitimate difference between these in the sense that a TRO is not isomorphic to an anti-TRO or vice-versa. This is related to the uniqueness in Theorem (3.1) of [**13**].

Fortunately we will only have to deal with TROs here, mainly because the fibers of a C^*-algebraic bundle are TROs, a fact that follows from [**4**, VIII.16.5].

Loosely following [**13**], and occasionally offering minor improvements, we propose to discuss below a few facts about TROs which will be needed in the sequel. We will often use the notation "$a \cdot b \cdot c$" in place of "ab^*c" because most of our results concern the intrinsic structure of TROs, irrespective of the Hilbert space representation which is, nevertheless, always in the background.

Let E be a TRO which we consider fixed for the time being.

4.3. Definition. *A map $T: E \to E$ is said to be a left (resp. right) operator if there exists another map $T^*: E \to E$ satisfying*

$$a \cdot T(b) \cdot c = a \cdot b \cdot T^*(c)$$

$$(\text{resp.} \quad a \cdot T(b) \cdot c = T^*(a) \cdot b \cdot c)$$

Note that, since the (ternary) multiplication on E is non-degenerate in view of (4.1.iii), T^*, if it exists, must be unique. The definition also implies that a left (resp. right) operator must necessarily be a bounded linear map (for boundedness, use the closed graph theorem).

4.4. Proposition. *If T is a left (resp. right) operator, then*

i) T^* is also a left (resp. right) operator and $T^{**} = T$.

ii) For any $a, b, c \in E$ one has $T(a) \cdot b \cdot c = T(a \cdot b \cdot c)$ (resp. $a \cdot b \cdot T(c) = T(a \cdot b \cdot c)$) .

Proof. Let us assume T is a left operator and let $a, b, c, x, y \in E$. Then

$$x \cdot (a \cdot T^*(b) \cdot c) \cdot y = (x \cdot c \cdot T^*(b)) \cdot a \cdot y$$

$$= (x \cdot T(c) \cdot b) \cdot a \cdot y = x \cdot (a \cdot b \cdot T(c)) \cdot y$$

This, together with the non-degeneracy of the product, implies that T^* is a left operator and that its adjoint is T.

To prove (ii) we have

$$x \cdot (T(a) \cdot b \cdot c) \cdot y = x \cdot c \cdot (b \cdot T(a) \cdot y) = x \cdot c \cdot (b \cdot a \cdot T^*(y))$$

$$= x \cdot (a \cdot b \cdot c) \cdot T^*(y) = x \cdot T(a \cdot b \cdot c) \cdot y.$$

The proof for right operators is similar. □

4.5. Proposition. *The set of all left (resp. right) operators on E is a C^*-algebra under the composition of operators, the involution defined above and the operator norm.*

Proof. Let T be a left operator. For x in E we have

$$\|T(x)\|^3 = \|T(x) \cdot T(x) \cdot T(x)\| = \|T(x) \cdot x \cdot T^*(T(x))\|$$

$$\leq \|T(x)\| \, \|x\| \, \|T^*(T(x))\| \leq \|T\| \, \|T^*T\| \, \|x\|^3.$$

This shows that $\|T\|^3 \leq \|T\| \, \|T^*T\|$ from which it follows that $\|T\|^2 \leq \|T^*T\|$. This can now be used to show both the norm preservation of the adjoint operation and the C^*-identity: $\|T\|^2 = \|T^*T\|$. The verification of the remaining properties is left to the reader. □

The algebra of left operators on E will be denoted $\mathcal{L}(E)$ and will be called the left algebra. Similarly we have the right algebra $\mathcal{R}(E)$. Please note that we are not using the same notation as in [**13**].

Given x and y in E, consider the maps $\lambda_{xy}: E \to E$ and $\rho_{xy}: E \to E$ defined by

$$\lambda_{xy}(a) = x \cdot y \cdot a \quad \text{and} \quad \rho_{xy}(a) = a \cdot x \cdot y$$

for all a in E. Note that for $a, b, c \in E$ we have

$$a \cdot \lambda_{xy}(b) \cdot c = a \cdot (x \cdot y \cdot b) \cdot c = a \cdot b \cdot (y \cdot x \cdot c) = a \cdot b \cdot \lambda_{yx}(c)$$

so λ_{xy} is a left operator and $\lambda_{xy}^* = \lambda_{yx}$. Similarly, ρ_{xy} is a right operator and $\rho_{xy}^* = \rho_{yx}$.

If T is a left operator, then one can easily show that

$$T\lambda_{xy} = \lambda_{T(x),y}$$

and that

$$\lambda_{xy}T = \lambda_{x,T^*(y)}.$$

So, one concludes that the closed linear span of the set of all λ_{xy}, within $\mathcal{L}(E)$, is an ideal which we denote by $E \otimes E^*$. Similarly $E^* \otimes E$ is the ideal of $\mathcal{R}(E)$ given by the closed linear span of the ρ_{xy}.

Observe that, as a consequence of $EE^*E \subseteq E$, one has that both EE^* and E^*E are closed under composition of operators.

Before we proceed, let us establish a slightly unusual notational convention which will, nevertheless, serve our purposes rather well:

4.6. Definition. *If X and Y are sets of elements such that some kind of multiplication xy is defined for x in X and y in Y, taking values in some normed linear space, then XY denotes the closed linear span of the set of products xy with $x \in X$ and $y \in Y$. This applies, in particular, to subsets of a C^*-algebra and also when X is a set of operators and Y is a set of vectors operated upon by the elements of X. The extreme situation in which X consists of a single element 1, which acts as a neutral element on Y, will be enforced as well. That is, $1Y$ is the closed linear span of Y rather than Y itself.*

So EE^* and E^*E, once interpreted according to the above definition, are actually C^*-algebras of operators on the Hilbert space where E acts. We would now like to prove that these are isomorphic to $E \otimes E^*$ and $E^* \otimes E$, respectively. This should be thought of as an indication that TROs are abstract objects which do not depend so much on the representation considered. The precise expression of this truth is Theorem (3.1) in [**13**], which we already mentioned.

4.7. Lemma. *Let E be a TRO on the Hilbert space H. If a is in EE^* and b is in E^*E then*

i) $\|a\| = \sup\{\|ax\|: x \in E, \|x\| \leq 1\}$

ii) $\|b\| = \sup\{\|xb\|: x \in E, \|x\| \leq 1\}$

Proof. We prove only (i). Define a norm $|||\cdot|||$ on EE^* using the right hand side of (i). Since E is invariant under left multiplication by EE^*, it follows that EE^* is a normed algebra under this new norm.

Note that for c in EE^* we have

$$|||c|||^2 = \sup_{\substack{\|x\|=1\\ x\in E}} \|cx\|^2 = \sup_{\substack{\|x\|=1\\ x\in E}} \|x^*c^*cx\| \le \sup_{\substack{\|x\|=1\\ x\in E}} \|c^*cx\| = |||c^*c|||$$

Therefore $|||c|||^2 \le |||c^*c|||$ which implies that EE^* is a pre-C^*-algebra under this norm. But, since $|||a||| \le \|a\|$ we must have $|||a||| = \|a\|$. Part (ii) follows similarly. □

4.8. Proposition. *Let E be a TRO on H. Then there are bijective C^*-algebra isomorphisms*

$$\phi\colon E \otimes E^* \to EE^* \quad \text{and} \quad \psi\colon E^* \otimes E \to E^*E$$

such that

$$\phi(\lambda_{xy}) = xy^* \quad \text{and} \quad \psi(\rho_{xy}) = x^*y.$$

Proof. Let $\alpha \in E \otimes E^*$ be the finite sum, $\alpha = \sum \lambda_{x_iy_i}$. Define $\phi(\alpha) = \sum x_iy_i^*$. To see that this is well defined note that, with the help of Lemma (4.7), we have

$$\|\sum x_iy_i^*\| = \sup_{\substack{\|z\|=1\\ z\in E}} \|\sum x_iy_i^*z\| =$$

$$\sup_{\substack{\|z\|=1\\ z\in E}} \|\sum x_i\cdot y_i\cdot z\| = \sup_{\substack{\|z\|=1\\ z\in E}} \|\sum \lambda_{x_iy_i}(z)\| = \|\alpha\|.$$

This shows that ϕ is well defined and isometric. The remaining verifications are left to the reader. □

Note that the essential spaces of EE^* and E^*E are, respectively, EH and E^*H. In addition the members of E should be thought of as being operators from E^*H to EH since they all vanish on the orthogonal complement of the former, and have their image contained in the latter. The following fact, which we will use frequently, has appeared in [**3**, Proposition 2.6].

4.9. Proposition. *If $\{u_i\}_i$ is an approximate identity for EE^* (resp. E^*E), then for all x in E we have $\lim_i u_ix = x$ (resp. $\lim_i xu_i = x$).*

It is a consequence of this, that:

4.10. Corollary. *If E is a TRO then $EE^*E = E$.*

Let us now study the question of stability for TROs.

4.11. Definition. *A TRO E is said to be left (resp. right) stable if $E\otimes E^*$ (resp. $E^*\otimes E$) is a stable C^*-algebra. In case E is both left and right stable, we simply say that E is stable.*

A simple example of a TRO which is left-stable but not right-stable is a Hilbert space equipped with the ternary multiplication $\xi \cdot \eta \cdot \zeta = \xi\langle\eta, \zeta\rangle$ (where we think of $\langle\cdot\,,\cdot\rangle$ as being conjugate-linear in the first variable).

In the following we let $\mathcal{K}$ denote the C^*-algebra of all compact operators on a separable infinite dimensional Hilbert space.

4.12. Proposition. *Suppose E has the structure of a left (resp. right) module over $\mathcal{K}$ such that for all $a, b, c \in \mathcal{K}$ and k in $\mathcal{K}$*

$$a \cdot (kb) \cdot c = a \cdot b \cdot (k^*c)$$

$$\textit{(resp.} \quad a \cdot (bk) \cdot c = (ak^*) \cdot b \cdot c).$$

Suppose further that $E = \mathcal{K}E$ (resp. $E = E\mathcal{K}$). Then E is left (resp. right) stable.

Proof. Suppose that E is a left $\mathcal{K}$-module. Then the left multiplication of elements of K on E gives a *-homomorphism $\mathcal{K} \to \mathcal{L}(E)$. Recall that

$$k\lambda_{xy} = \lambda_{kx,y}$$

which implies that $\mathcal{K}(E \otimes E^*) \subseteq E \otimes E^*$. Also, in view of the fact that $\mathcal{K}E = E$, and the equation above, we actually conclude that $\mathcal{K}(E \otimes E^*) = E \otimes E^*$.

It can be shown without much difficulty that if A is a C^*-subalgebra of another C^*-algebra, which also contains a copy of the compact operators $\mathcal{K}$, such that $\mathcal{K}A = A$, then A must be stable. Since this is precisely the case for $E \otimes E^*$, it follows that this algebra is stable. The case of right $\mathcal{K}$-modules is treated similarly. □

5. Regular TROs

Given a TRO E, note that E has a bi-module structure with respect to the algebra $\mathcal{L}(E)$ acting on the left and $\mathcal{R}(E)$, on the right (this means, in particular, that a left operator commutes with any right operator [**13**, 3.4]). In addition, E is an imprimitivity bi-module for the ideals $E \otimes E^* \subseteq \mathcal{L}(E)$ and $E^* \otimes E \subseteq \mathcal{R}(E)$ [**9**], [**10**], [**11**] with inner-products defined by

$$(x, y) \in E \to (x|y) = \lambda_{xy} \in E \otimes E^*$$

and

$$(x, y) \in E \to \langle x, y\rangle = \rho_{xy} \in E^* \otimes E.$$

The very delicate point of whether these inner products are positive is not an issue here, precisely because we are dealing exclusively with TROs, as opposed to general ternary C^*-rings. In fact, under the identification of $E \otimes E^*$ and EE^*, we have that $(x|x) = xx^*$ which is obviously a positive operator in EE^*. Likewise $\langle x, x\rangle = x^*x$ is positive.

Closely associated with the notion of imprimitivity bi-modules, there is the concept of linking algebra. Recall that the linking algebra [**1**], [**2**] is the C^*-algebra

$$\mathrm{Link}(E) = \begin{pmatrix} E \otimes E^* & E \\ E^* & E^* \otimes E \end{pmatrix}$$

equipped with the multiplication

$$\begin{pmatrix} a_1 & x_1 \\ y_1^* & b_1 \end{pmatrix} \begin{pmatrix} a_2 & x_2 \\ y_2^* & b_2 \end{pmatrix} = \begin{pmatrix} a_1 a_2 + (x_1|y_2) & a_1 x_2 + x_1 b_2 \\ y_1^* a_2 + b_1 y_2^* & \langle y_1, x_2 \rangle + b_1 b_2 \end{pmatrix}$$

and involution

$$\begin{pmatrix} a & x \\ y^* & b \end{pmatrix}^* = \begin{pmatrix} a^* & y \\ x^* & b^* \end{pmatrix}$$

for $a, a_1, a_2 \in E \otimes E^*$, $b, b_1, b_2 \in E^* \otimes E$ and $x, x_1, x_2, y, y_1, y_2 \in E$.

Recall from [**1**] that two C^*-algebras are said to be strongly Morita equivalent to each other if there exists an imprimitivity bi-module. Of course, whenever E is a TRO, the algebras $E \otimes E^*$ and $E^* \otimes E$ are Morita equivalent to each other. If we add to this situation the hypothesis that E is stable and that both $E \otimes E^*$ and $E^* \otimes E$ possess strictly positive elements, then the well known result of Brown, Green and Rieffel [**BGR**, Theorem 1.2] implies that $E \otimes E^*$ and $E^* \otimes E$ are isomorphic C^*-algebras. To pinpoint the precise consequence of this circle of ideas that we will need is the main goal of the present section.

In the following we let E be a fixed TRO.

5.1. Definition. *We say that E is regular if there exists a partial isometry v in the multiplier algebra of* $\mathrm{Link}(E)$ *such that*

$$vv^* = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \textit{and} \quad v^*v = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

(Compare Lemma (3.3) of [**1**]*).*

5.2. Proposition. *Any separable stable TRO is regular.*

Proof. This is an immediate consequence of combining (3.4) and (3.3) in [**1**], as long as we note that separability of E implies separability of both $E \otimes E^*$ and $E^* \otimes E$ and hence the existence of strictly positive elements. ◻

The following characterizes regular TROs at the Hilbert space level.

5.3. Proposition. *Let E be a TRO on H. Then E is regular if and only if there exists a partially isometric operator u on H such that*

$$uE^* = EE^* \quad \text{and} \quad u^*E = E^*E.$$

(Here as everywhere else in this work, we keep (4.6) in force. In particular uE^*, u^*E, EE^* and E^*E are all meant to denote the closed linear span of the set of products).

Proof. Assume that E is regular and hence assume the existence of $v \in \mathcal{M}(\mathrm{Link}(E))$ as above. Denote by $e_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $e_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, both of which are viewed as elements in $\mathcal{M}(\mathrm{Link}(E))$. We have

$$v^* \begin{pmatrix} 0 & E \\ 0 & 0 \end{pmatrix} = v^* e_1 \mathrm{Link}(E) e_2 = v^* v v^* \mathrm{Link}(E) e_2 =$$

$$e_2 v^* \mathrm{Link}(E) e_2 \subseteq e_2 \mathrm{Link}(E) e_2 = \begin{pmatrix} 0 & 0 \\ 0 & E^* \otimes E \end{pmatrix} = \begin{pmatrix} 0 & E \\ 0 & 0 \end{pmatrix}^* \begin{pmatrix} 0 & E \\ 0 & 0 \end{pmatrix}.$$

Conversely

$$\begin{pmatrix} 0 & E \\ 0 & 0 \end{pmatrix}^* \begin{pmatrix} 0 & E \\ 0 & 0 \end{pmatrix} = e_2 \mathrm{Link}(E) e_2 = v^* v v^* v \mathrm{Link}(E) e_2 \subseteq v^* v v^* \mathrm{Link}(E) e_2$$

$$= v^* e_1 \mathrm{Link}(E) e_2 = v^* \begin{pmatrix} 0 & E \\ 0 & 0 \end{pmatrix}.$$

This proves that

$$v^* \begin{pmatrix} 0 & E \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & E \\ 0 & 0 \end{pmatrix}^* \begin{pmatrix} 0 & E \\ 0 & 0 \end{pmatrix}$$

and in a similar way we could show that

$$v \begin{pmatrix} 0 & E \\ 0 & 0 \end{pmatrix}^* = \begin{pmatrix} 0 & E \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & E \\ 0 & 0 \end{pmatrix}^*.$$

Consider the representation of $\mathrm{Link}(E)$ on $H \oplus H$ given by interpreting an element $\begin{pmatrix} a & x \\ y^* & b \end{pmatrix}$ as an operator on $H \oplus H$ in the obvious way. Let u be the image of v under the canonic extension of that representation to $\mathcal{M}(\mathrm{Link}(E))$. Since $vv^* = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $v^*v = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ it follows that u must actually have the form $\begin{pmatrix} 0 & u \\ 0 & 0 \end{pmatrix}$, where $u \colon H \to H$ is a partially isometric operator.

This said we see that the image of $v^* \begin{pmatrix} 0 & E \\ 0 & 0 \end{pmatrix}$ in $\mathcal{B}(H \oplus H)$ will thus be

$$\begin{pmatrix} 0 & 0 \\ u^* & 0 \end{pmatrix} \begin{pmatrix} 0 & E \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & u^* E \end{pmatrix}$$

which will coincide, by what we saw above, with $\begin{pmatrix} 0 & 0 \\ 0 & E^* E \end{pmatrix}$. This shows that $u^* E = E^* E$. Similarly $uE^* = EE^*$.

To prove the converse statement, one defines v to be the multiplier on $\operatorname{Link}(E)$ whose left and right actions are given by multiplying on the left and right by the operator $\begin{pmatrix} 0 & u \\ 0 & 0 \end{pmatrix}$. Although

$$\begin{pmatrix} 0 & u \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & u \\ 0 & 0 \end{pmatrix}^* = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

may not hold as operators on $H \oplus H$, that equality is true as long as multipliers of $\operatorname{Link}(E)$ are concerned. Similarly $v^* v = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. □

5.4. Definition. *If E is a TRO and u is a partial isometry such that $uE^* = EE^*$ and $u^* E = E^* E$, we say that u is associated to E and we write $u \sim E$.*

There is no obvious sense in which a partial isometry associated to E is unique. In particular, the equations $uE^* = EE^*$ and $u^* E = E^* E$ do not even determine, in general, the initial and final space of u. However we can at least affirm that the initial space of such a u contains $E^* H$. In fact

$$E^* H = E^* E E^* H = u^* E E^* H \subseteq u^* H.$$

Likewise we have that EH is contained in the image of u. Another conclusion we can draw from the fact that $u \sim E$, is that u defines an isometry from $E^* H$ to EH. To see this note that $uE^* H = EE^* H \subseteq EH$ while $u^* EH = E^* EH \subseteq E^* H$.

5.5. Definition. *Let $u \sim E$. Then u is said to be strictly associated to E if the initial space of u coincides with $E^* H$. Equivalently, if the final space of u is EH. In this case we write $u \overset{s}{\sim} E$.*

Observe that, in case $u \sim E$, but not strictly, then we can make it strict by replacing u by up where p is the orthogonal projection onto $E^* H$. The property of being associated to E will not notice that change.

A strict partial isometry also has a topological relationship to E:

5.6. Proposition. *Let E ba a TRO on H and assume that the partial isometry u is associated to E. Then a necessary and sufficient condition for u to be strictly associated to E is that u be in the strong operator closure of E within $\mathcal{B}(H)$.*

Proof. Let $\{e_i\}_i$ be an approximate identity for EE^*. Then it is well known that e_i converges strongly to the the orthogonal projection onto the essential space of EE^*, which we have seen to coincide with EH. Let $u_i = e_i u$. Then $u_i \in EE^* u = EE^* E = E$. Observing that the range of u is EH, we conclude that u_i converges strongly to u. Conversely, if u is in the strong operator closure of E, then it must vanish on the orthogonal complement of $E^* H$, as is the case for any member of E. This concludes the proof. □

Returning to our earlier discussion on the question of uniqueness for a partial isometry associated to E observe that even strict ones are not unique. In fact if $u \overset{s}{\sim} E$ and if w is a unitary multiplier of EE^*, then $wu \overset{s}{\sim} E$.

With respect to the nature of the product wu above, we need to clarify the following point. If A is a C^*-algebra which is represented on a Hilbert space H under a faithful *non-degenerate* representation, then it is well known [**7**, 3.12.3] that its multiplier algebra $\mathcal{M}(A)$ can also be represented within $\mathcal{B}(H)$. However, if that representation is not non-degenerated, i.e, if the essential space AH is a proper subspace of H, then this is still true in the sense that $\mathcal{M}(A)$ is isomorphic to the algebra consisting of those operators m in $\mathcal{B}(H)$ such that both mA and Am are contained in A and such that both m and m^* vanish on the orthogonal complement of AH. Therefore, when we spoke of w above, we meant an operator on H and hence wu should be simply interpreted as the composition of operators.

5.7. Proposition. *Let u_1 and u_2 be strict partial isometries associated to E. Then $u_2u_1^*$ is a unitary multiplier in $\mathcal{M}(EE^*)$ and $u_2^*u_1$ is a unitary multiplier in $\mathcal{M}(E^*E)$.*

Proof. Proving the first statement amounts to verifying that $u_2u_1^*$ is a unitary operator on EH and that EE^* is invariant under both left and right multiplication by $u_2u_1^*$, all of which follow by routine arguments. ☐

Still under the notation above, note that if $w = u_2u_1^*$, then $u_2 = wu_1$ and so we see that strict partial isometries are unique, after all, up to multiplication by a unitary element in $\mathcal{M}(EE^*)$.

6. Ideals of TROs

The classification of C^*-algebraic bundles we are about to discuss requires a careful understanding of the relationship between TROs and its subspaces, specially when these are ideals in the sense below.

6.1. Definition. *Let J be a closed subspace of the TRO E. Then J is said to be an ideal if*

$$J \cdot J \cdot E \subseteq J \quad \textit{and} \quad E \cdot J \cdot J \subseteq J.$$

We remark that there is a total number of eight possible ways of combining E and J in the ternary product, so there are many alternatives to the definition of the concept of ideals in TROs. Even though we don't claim to have experimented with too many of those, we hope to convince the reader that our choice is meaningful.

Note that $J \cdot J \cdot E$ above, means JJ^*E and a similar remark applies to $E \cdot J \cdot J$. From now on we will use the latter notation.

6.2. Lemma. *If J is an ideal in E, then*

i) $JJ^*E = J$

ii) $EJ^*J = J$

iii) $JJ^*EE^* = JJ^*$.

iv) $J^*JE^*E = J^*J$.

Proof. Initially observe that J is a TRO in its own right. With respect to (i) we have

$$J = JJ^*J \subseteq JJ^*E \subseteq J$$

so $J = JJ^*E$. The proof of (ii) is similar. As for (iii) we have, using (i) and (ii)

$$JJ^*EE^* = JE^* = JJ^*JE^* = J(EJ^*J)^* = JJ^*.$$

We leave (iv) to the reader. □

6.3. Proposition. *An ideal J of a regular TRO E is necessarily regular. In addition if u is a partial isometry with $u \sim E$, then $u \sim J$.*

Proof. It obviously suffices to prove the second assertion. For that purpose we use (6.2) in the following calculations

$$uJ^* = uE^*JJ^* = EE^*JJ^* = JJ^*$$

and

$$u^*J = u^*EJ^*J = E^*EJ^*J = J^*J.$$ □

Another fact we need for future use is proven below.

6.4. Proposition. *Let E and F be TROs on H, such that $FF^*E^*E = E^*EFF^*$, then*

i) EF *is a TRO.*

ii) If $u \overset{s}{\sim} E$ *and* $v \overset{s}{\sim} F$ *then* $uv \overset{s}{\sim} EF$.

Proof. That EF is a TRO follows from the following calculation

$$EFF^*E^*EF = EE^*EFF^*F = EF.$$

To prove (ii) we first claim that the final projection vv^* of v commutes with the initial projection u^*u of u. To see this it is enough to show that the range of vv^*, which coincides with FH, is invariant under u^*u. With that goal in mind note that $uFH \subseteq EFH$ because u is in the strong closure of E. Hence

$$u^*u(FH) \subseteq u^*EFH = E^*EFF^*FH = FF^*E^*EFH \subseteq FH.$$

This shows our claim that u^*u and vv^* commute and hence that uv is a partial isometry. To show that $uv \sim EF$ we compute

$$uvF^*E^* = uFF^*E^*EE^* = uE^*EFF^*E^* = EFF^*E^*.$$

That $(uv)^*EF = (EF)^*EF$ follows similarly. It now remains to show that the final space of uv is EFH. We have

$$uvH = uFH \subseteq EFH$$

again because u is in the strong closure of F. The opposite inclusion follows from the argument preceding (5.5). □

Our motivation for conducting a study of TROs is, of course, our interest in C^*-algebraic bundles. If $\mathcal{B}$ is a C^*-algebraic bundle and if r and s are elements in the base group, then it can be proven that B_rB_s is an ideal in B_{rs}. If everything is regular and represented in a Hilbert space, then we will have isometries u_r, u_s and u_{rs}. To understand the relationship between u_ru_s and u_{rs} is our last objective before we plunge into the main section of this work.

6.5. Proposition. *Suppose E, F and M are regular TROs such that*

i) $FF^*E^*E = E^*EF^*F$

ii) EF *is an ideal in* M.

Let u, v and z be partial isometries strictly associated to E, F and M, respectively. Then uvz^* is a unitary multiplier of EFF^*E^*.

Proof. We know from (6.4) that $uv \overset{s}{\sim} EF$. On the other hand (6.3) tells us that also $z \sim EF$ although possibly not strictly. But if p is the orthogonal projection onto $(EF)^*H$, then $zp \overset{s}{\sim} EF$. It then follows from (5.7) that the operator w defined by $w = uv(zp)^*$ is a unitary multiplier in $\mathcal{M}(EFF^*E^*)$. Finally observe that since uv is strict, then we must have $uvp = uv$ and hence $w = uvz^*$. ◻

7. The Classification of Stable C^*-Algebraic Bundles

Let $\mathcal{B}$ be a C^*-algebraic bundle over the locally compact group G, considered fixed throughout this section. Our goal is to show, upon assuming a certain regularity property of $\mathcal{B}$, that it is isomorphic to the semidirect product bundle constructed from a suitable twisted partial action of the base group G on the unit fiber algebra B_e. Construction of that action will be done in several steps.

Let us initially deal with the problem of defining the family $\{D_t\}_{t\in G}$ of ideals of B_e. We simply let, for each t in G, $D_t = B_tB_t^*$. Clearly D_t is an ideal in B_e. To see that the D_t form a continuous family, recall from (3.3) that this follows once we provide a pointwise-dense set of sections. Now observe that if γ and δ are in $\mathcal{C}_0(\mathcal{B})$, the space of continuous sections of $\mathcal{B}$ vanishing at infinity [**4**, II.14.7], then $\gamma(t)\delta(t)^*$ is a continuous B_e valued function which satisfies $\gamma(t)\delta(t)^* \in D_t$ for all t. In other words it is a continuous section for the family $\{D_t\}_{t\in G}$. The linear span of the set of such sections is clearly pointwise-dense and hence this proves continuity.

So far we have thus been able to construct a Banach bundle over G from the D_t's, according to (3.2). Let us denote the total space of this bundle by $\mathcal{D}$.

Recall from [**4**, II.14.1] that $\mathcal{C}_0(\mathcal{B})$ is a Banach space under the supremum norm. It matters to us that it is also a TRO under the ternary operation $(\gamma\cdot\delta\cdot\varepsilon)(t) = \gamma(t)\delta(t)^*\varepsilon(t)$. Precisely, we mean that $\mathcal{C}_0(\mathcal{B})$ is isomorphic to a TRO in some Hilbert space, as far as its Banach space structure and the operation mentioned above are concerned. Let us now show how to represent $\mathcal{C}_0(\mathcal{B})$ as a TRO.

Let ρ be a representation of $\mathcal{B}$ in the sense of [**4**, VIII.8.2, VIII.9.1] such that the restriction of ρ to each B_t is isometric. The existence of such a representation follows from [**4**, VIII.16.5]. We may therefore assume that each B_t is a closed subspace of $\mathcal{B}(H)$ for some Hilbert space H (which does not depend on t) and moreover we have $B_rB_s \subseteq B_{rs}$ and $B_t^* = B_{t^{-1}}$. In particular this implies that each B_t is a TRO. Denoting by $l_2(G)$ the Hilbert space of all square summable sequences of complex numbers, indexed by G (we are temporarily ignoring the topology of G here), consider the map

$$\pi\colon \mathcal{C}_0(\mathcal{B}) \to \mathcal{B}\left(H \otimes l_2(G)\right)$$

given by $\pi(\gamma)(\xi \otimes e_t) = (\gamma(t)\xi) \otimes e_t$ for all γ in $\mathcal{C}_0(\mathcal{B})$, ξ in H and t in G, where we are denoting by e_t the canonic basis of $l_2(G)$. In other words $\pi(\gamma)$ is the diagonal operator $\operatorname{diag}(\gamma(t)_{t\in G})$, with respect to the decomposition of $H \otimes l_2(G)$ provided by the canonic basis of $l_2(G)$. Clearly π is an isometric representation of $\mathcal{C}_0(\mathcal{B})$ which satisfies

$$\pi(\gamma \cdot \delta \cdot \varepsilon) = \pi(\gamma)\pi(\delta)^*\pi(\varepsilon).$$

In other words we may identify $\mathcal{C}_0(\mathcal{B})$ with its image in $\mathcal{B}\left(H \otimes l_2(G)\right)$ through π. This identification will be tacitly made henceforth, without explicit mention to π.

In order to be able to apply the machinery of regular TROs developed above, we would like to have $\mathcal{C}_0(\mathcal{B})$ regular. However this is not within our reach unless we make extra requirements.

7.1. Proposition. *If B_e is a stable C^*-algebra then $\mathcal{C}_0(\mathcal{B})$ is stable as a TRO. If, in addition, $\mathcal{B}$ is second countable, then $\mathcal{C}_0(\mathcal{B})$ is regular.*

Proof. There is an obvious way in which $\mathcal{C}_0(\mathcal{B})$ can be considered as a B_e bi-module. With a little more effort we can give $\mathcal{C}_0(\mathcal{B})$ the structure of a bi-module over the multiplier algebra $\mathcal{M}(B_e)$. In order to do this we use [**4**, VIII.3.8 and VIII.16.3] to identify the multiplier algebra of B_e with the set of multipliers of $\mathcal{B}$ of order e (see [**4**, VIII.2.14]). Thus, if γ is in $\mathcal{C}_0(\mathcal{B})$ and m is in $\mathcal{M}(B_e)$, we define $m\gamma$ in $\mathcal{C}_0(\mathcal{B})$ by $(m\gamma)(t) = m(\gamma(t))$ and likewise $(\gamma m)(t) = (\gamma(t))m$.

That $m\gamma$ and γm are continuous follows from the fact that the left and right action of m is a continuous bundle map from $\mathcal{B}$ to itself. The latter, in turn, follows from [**4**, II.13.16] with the set $\Gamma = \{b_1\gamma b_2 \in \mathcal{C}_0(\mathcal{B})\colon b_1, b_2 \in B_e, \gamma \in \mathcal{C}_0(\mathcal{B})\}$. The reason why Γ is pointwise dense, finally follows from the existence of approximate identities [**4**, VIII.16.3].

Observe that these module structures are compatible with the *-operation in the sense that

$$\gamma \cdot m\delta \cdot \varepsilon = \gamma \cdot \delta \cdot m^*\varepsilon \quad \text{and} \quad \gamma \cdot \delta m \cdot \varepsilon = \gamma m^* \cdot \delta \cdot \varepsilon.$$

In other words we have *-homomorphisms

$$\Lambda\colon \mathcal{M}(B_e) \to \mathcal{L}(\mathcal{C}_0(\mathcal{B})))$$

and

$$\mathrm{P}\colon \mathcal{M}(B_e) \to \mathcal{R}(\mathcal{C}_0(\mathcal{B})).$$

Given that B_e is stable, say $B_e = A \otimes \mathcal{K}$ for some C^*-algebra A, consider the left and right actions of K on B_e given by $k_1(a \otimes k_2) := a \otimes (k_1 k_2)$ and $(a \otimes k_2)k_1 := a \otimes (k_2 k_1)$ for $k_1, k_2 \in \mathcal{K}$ and $a \in A$. These satisfy $\mathcal{K}B_e = B_e\mathcal{K} = \mathcal{B}_e$ and define a *-homomorphism $\phi: \mathcal{K} \to \mathcal{M}(B_e)$. If we now follow this map by either Λ or P above, we will give $\mathcal{C}_0(\mathcal{B})$ the structure of a $\mathcal{K}$ bi-module. We must now prove that $\mathcal{K}\mathcal{C}_0(\mathcal{B}) = \mathcal{C}_0(\mathcal{B})\mathcal{K} = \mathcal{C}_0(\mathcal{B})$ to be in condition to apply (4.12). This follows from

$$\mathcal{K}\mathcal{C}_0(\mathcal{B}) = \mathcal{K}B_e\mathcal{C}_0(\mathcal{B}) = B_e\mathcal{C}_0(\mathcal{B}) = \mathcal{C}_0(\mathcal{B})$$

and the corresponding right hand sided version.

As for the second assertion in the statement, assume that $\mathcal{B}$ is second countable. Then [**4**, II.14.10] tells us that the space of compactly supported continuous sections is separable in the inductive limit topology. From this we can then deduce that $\mathcal{C}_0(\mathcal{B})$ is separable in the sup norm. The conclusion is thus reached, upon invoking (5.2). □

Although we will not explicitly need it, stability of B_e implies stability of each fiber B_t as well.

Let us assume, from now on, that $\mathcal{B}$ is a C^*-algebraic bundle for which $\mathcal{C}_0(\mathcal{B})$ is a regular TRO. Of course this will include all second countable C^*-algebraic bundles for which the unit fiber algebra is stable.

As mentioned earlier in this section, for each pair of sections γ and δ in $\mathcal{C}_0(\mathcal{B})$, we have that $\gamma\delta^*$ is a section of $\mathcal{C}_0(\mathcal{D})$. Note that $\mathcal{C}_0(\mathcal{D})$ can also be represented in $H \otimes l_2(G)$ via diagonal operators. Under these representations we therefore have $\mathcal{C}_0(\mathcal{B})\mathcal{C}_0(\mathcal{B})^* \subseteq \mathcal{C}_0(\mathcal{D})$. Using [**4**, II.14.7] we actually obtain $\mathcal{C}_0(\mathcal{B})\mathcal{C}_0(\mathcal{B})^* = \mathcal{C}_0(\mathcal{D})$.

Recall that $\mathcal{D}^{-1}$ denotes the Banach bundle over G which is obtained by placing the ideal $D_{t^{-1}}$ as the fiber over t, according to (3.2). Representing $\mathcal{C}_0(\mathcal{D}^{-1})$ also via diagonal operators (each $D_{t^{-1}}$ acting on $H \otimes e_t$) we will have, by a similar reasoning, that $\mathcal{C}_0(\mathcal{B})^*\mathcal{C}_0(\mathcal{B}) = \mathcal{C}_0(\mathcal{D}^{-1})$.

Since we are assuming that $\mathcal{C}_0(\mathcal{B})$ is regular, we may invoke (5.3) to conclude that there must exist a partial isometry U in $\mathcal{B}(H \otimes l_2(G))$ which is strictly associated with $\mathcal{C}_0(\mathcal{B})$. By (5.6) we conclude that U is in the strong closure of $\mathcal{C}_0(\mathcal{B})$ within $\mathcal{B}(H \otimes l_2(G))$ which, in turn, implies that U must be a diagonal operator. That is, $U = \operatorname{diag}((u_t)_{t \in G})$, where, for for each t in G, u_t is a partial isometry in $\mathcal{B}(H)$. Upon replacing u_t by $u_e^* u_t$ we may assume that $u_e = 1$.

Expressing in formulas the fact that $U \overset{s}{\sim} \mathcal{C}_0(\mathcal{B})$ we have

$$U\mathcal{C}_0(\mathcal{B})^* = \mathcal{C}_0(\mathcal{B})\mathcal{C}_0(\mathcal{B})^* = \mathcal{C}_0(\mathcal{D})$$

and

$$U^*\mathcal{C}_0(\mathcal{B}) = \mathcal{C}_0(\mathcal{B})^*\mathcal{C}_0(\mathcal{B}) = \mathcal{C}_0(\mathcal{D}^{-1}).$$

Under point evaluation at each group element t (meaning to focus on a specific diagonal entry) we see that each $u_t \sim B_t$. Moreover, since U is strict with respect to $\mathcal{C}_0(\mathcal{B})$ we can easily prove that u_t is strict with respect to B_t.

Given that $D_t = B_t B_t^* = B_t u_t^*$, we may define, for each t in G, a map

$$b \in B_t \to b u_t^* \in D_t$$

which is clearly an isometry onto D_t and hence provides a bundle map

$$\rho^\dagger \colon \mathcal{B} \to \mathcal{D}.$$

That $\rho^\dagger$ is continuous follow from [**4**, II.13.16] and the remark that for any continuous section γ of $\mathcal{B}$ one has that

$$\rho^\dagger(\gamma(t)) = \gamma(t) u_t^* = (\gamma U^*)(t)$$

which is a member of $\mathcal{C}_0(\mathcal{B})\mathcal{C}_0(\mathcal{B})^* = \mathcal{C}_0(\mathcal{D})$ and hence is continuous. It now follows from [**4**, II.13.17] that $\rho^\dagger$ is an isometric isomorphism of Banach bundles. Its inverse is clearly the map

$$\rho \colon \mathcal{D} \to \mathcal{B}$$

given by gluing together the maps

$$a \in D_t \to a u_t \in B_t.$$

In an entirely similar way we have the isometric Banach bundle isomorphism

$$\lambda^\dagger \colon \mathcal{B} \to \mathcal{D}^{-1}$$

which, together with its inverse

$$\lambda \colon \mathcal{D}^{-1} \to \mathcal{B}$$

are given by

$$\lambda^\dagger \colon b \in B_t \to u_t^* b \in D_{t^{-1}} \quad \text{and} \quad \lambda \colon b \in D_{t^{-1}} \to u_t b \in B_t.$$

Now define, for each x in $D_{t^{-1}}$,

$$\theta_t(x) = u_t x u_t^*.$$

Note that

$$\theta_t(D_{t^{-1}}) = \theta_t(B_t^* B_t) = u_t B_t^* B_t u_t^* = B_t B_t^* = D_t.$$

Since $u_t^* u_t$ is the identity on $B_t^* H$, we see that θ_t is a C^*-algebra isomorphism. The continuity of θ, in the sense of (3.5), is now obvious since the corresponding bundle map $\theta \colon \mathcal{D}^{-1} \to \mathcal{D}$ is just the composition

$$\mathcal{D}^{-1} \xrightarrow{\lambda} \mathcal{B} \xrightarrow{\rho^\dagger} \mathcal{D}.$$

The only missing ingredient of our twisted partial action is now the multipliers $w(r,s)$ mentioned in (2.1). Referring to the notation of (6.5), let, for each r and s in G, $E = B_r$, $F = B_s$ and $M = B_{rs}$. Once one verifies that the hypothesis of (6.5) are verified, we will conclude that the element $w(r,s)$ defined by $w(r,s) = u_r u_s u_{rs}^*$ is a unitary multiplier of $B_r B_s B_s^* B_r^*$. As for (6.5.i), this holds because both $B_s B_s^*$ and $B_r^* B_r$ are ideals in B_e and hence their product in either order coincides with their intersection.

7.2. Lemma. *For all r and s in G one has $B_rB_sB_s^*B_r^* = D_r \cap D_{rs}$.*

Proof. We have

$$B_rB_sB_s^*B_r^* = B_rB_r^*B_rB_sB_s^*B_r^* \subseteq B_rB_r^*B_{rs}B_{rs}^* = D_r \cap D_{rs}.$$

Conversely

$$D_r \cap D_{rs} = D_rD_{rs} = D_rD_{rs}D_r = B_rB_r^*B_{rs}B_{rs}^*B_rB_r^* \subseteq B_rB_sB_s^*B_r^*. \qquad \square$$

With this at hand we see that $w(r,s)$ is a multiplier of $D_r \cap D_{rs}$ as it is called for by (2.1). To see that $w(r,s)$ is continuous as a section of the bundle formed by the $D_r \cap D_{rs}$ we can employ [**4**, II.14.16] using the pointwise-dense space of sections that is spanned by the sections of the form $(r,s) \mapsto \gamma(r)\delta(s)\varepsilon(s)^*\zeta(r)^*$ where $\gamma, \delta, \varepsilon$ and ζ are in $\mathcal{C}_0(\mathcal{B})$.

The following is our main result:

7.3. Theorem. *Let G be a locally compact group and let $\mathcal{B}$ be a C^*-algebraic bundle over G such that $\mathcal{C}_0(\mathcal{B})$ is regular (e.g. if $\mathcal{B}$ is second countable and B_e is stable). Then, there exists a continuous twisted partial action*

$$\Theta = \big(\{D_t\}_{t\in G}, \{\theta_t\}_{t\in G}, \{w(r,s)\}_{(r,s)\in G\times G}\big)$$

of G on the unit fiber algebra B_e, such that $\mathcal{B}$ is isometricaly isomorphic to the semidirect product bundle of A and G constructed from Θ.

Proof. Define D_t, θ_t and $w(r,s)$ as above, and let us prove that

$$\Theta = \big(\{D_t\}_{t\in G}, \{\theta_t\}_{t\in G}, \{w(r,s)\}_{(r,s)\in G\times G}\big)$$

is a continuous twisted partial action. Starting with (2.1.b) we have

$$\theta_r(D_{r^{-1}} \cap D_s) = \theta_r(B_r^*B_rB_sB_s^*) = u_rB_r^*B_rB_sB_s^*B_r^*B_ru_r^* =$$

$$B_rB_sB_s^*B_r^* = D_r \cap D_{rs}.$$

To prove (2.1.c) let a be in $D_{s^{-1}} \cap D_{s^{-1}r^{-1}}$. Then

$$w(r,s)\theta_{rs}(a)w(r,s)^* = u_ru_su_{rs}^*u_{rs}au_{rs}^*u_{rs}u_s^*u_r^*.$$

Now, since $u_{rs}^*u_{rs}$ is the projection on E_{rs}^*H, it follows that $u_{rs}^*u_{rs}$ behaves like the identity element when it is multiplied by elements in $E_{rs}^*E_{rs} = D_{s^{-1}r^{-1}}$. This proves (2.1.c).

With respect to (2.1.d), recall that $u_e = 1$ so that

$$w(t,e) = u_tu_eu_t^* = u_tu_t^*$$

which is precisely the unit in the multiplier algebra $\mathcal{M}(D_t)$, at least according to our convention discussed before (5.7). Similarly $w(e,t)=1$.

The last axiom in (2.1) translates to the following, for $a \in D_{r^{-1}} \cap D_s \cap D_{st}$

$$u_r a u_s u_t u_{st}^* u_r^* u_r u_{st} u_{rst}^* = u_r a u_r^* u_r u_s u_{rs}^* u_{rs} u_t u_{rst}^*.$$

To see that this holds, all we need is to show that the initial projections of the various partial isometries appearing in this expression may be canceled out. This can be done by observing that, in all cases, that projection appears besides an operator which 'lives' in its range.

Now, given that the continuity of the various ingredients of our twisted partial action have already been verified, we conclude that Θ is indeed a continuous twisted partial action.

Let, therefore, $\mathcal{D}$ be the C^*algebraic bundle obtained from Θ as described in (3.10). To conclude we must prove that $\mathcal{B}$ and $\mathcal{D}$ are isomorphic C^*-algebraic bundle. Recall that we have already found a isometric Banach bundle isomorphism

$$\rho: \mathcal{D} \to \mathcal{B}$$

given by

$$\rho(a_t\delta_t) = a_t u_t, , \quad a_t \in D_t,$$

which we now claim to be a C^*-algebraic bundle isomorphism as well. To prove this claim all we need to check is that ρ is multiplicative and *-preserving. For a_r in D_r and b_s in D_s let us prove that

$$\rho(a\delta_r * b_s\delta_s) = \rho(a_r\delta_r)\rho(b_s\delta_s).$$

The left hand side equals

$$\rho(\theta_r\left(\theta_r^{-1}(a_r)b_s\right)w(r,s)\delta_{rs}) = \theta_r\left(\theta_r^{-1}(a_r)b_s\right)u_r u_s u_{rs}^* u_{rs} =$$

$$u_r(u_r^* a_r u_r b_s)u_r^* u_r u_s u_{rs}^* u_{rs} = a_r u_r b_s u_s = \rho(a_r\delta_r)\rho(a_s\delta_s).$$

Finally

$$\rho\left((a_r\delta_r)^*\right) = \rho\left(\theta_r^{-1}(a_r^*)w(r^{-1},r)^*\delta_{r^{-1}}\right) = u_r^* a_r^* u_r u_r^* u_{r^{-1}}^* u_{r^{-1}} = u_r^* a_r^* = (a_r u_r)^*. \qquad \square$$

References

[1] L. G. Brown, P. Green and M. A. Rieffel, "Stable isomorphism and strong Morita equivalence of C^*-algebras", *Pacific J. Math.* **71** (1977), 349–363.

[2] L. G. Brown, J. A. Mingo and N. T. Shen, "Quasi-multipliers and embeddings of Hilbert C^*-bimodules", preprint, Queen's University, 1992.

[3] R. Exel, "Circle actions on C^*-algebras, partial automorphisms and a generalized Pimsner–Voiculescu exact sequence", *J. Funct. Analysis*, to appear.

[4] J. M. G. Fell and R. S. Doran, "Representations of *-algebras, locally compact groups, and Banach *-algebraic bundles", Academic Press, Pure and Applied Mathematics, vols. 125 and 126, 1988.

[5] K. McClanahan, "K-theory for partial crossed products by discrete groups", preprint, University of Mississipi.

[6] J. A. Packer and I. Raeburn, "Twisted products of C^*-algebras", *Math. Proc. Camb. Phil. Soc* **106** (1989), 293–311.

[7] G. K. Pedersen, "C^*-Algebras and their Automorphism Groups", Academic Press, 1979.

[8] J. C. Quigg, "Discrete C^*-coactions and C^*-algebraic bundle", preprint, Arizona State University.

[9] M. A. Rieffel, "Induced representations of C^*-algebras", *Adv. Math.* **13** (1974), 176–257.

[10] M. A. Rieffel, "Morita equivalence for C^*-algebras and W^*-algebras", *J. Pure Appl. Algebra* **5** (1974), 51–96.

[11] M. A. Rieffel, "Strong Morita equivalence of certain transformation group C^*-algebras", *Math. Ann.* **222** (1976), 7–22.

[12] G. Zeller-Meier, "Produits croisés d'une C^*-algèbre par un group d'automorphismes", *J. Math Pures Appl.* **47** (1968), 101–239.

[13] H. Zettl, "A characterization of ternary rings of operators", *Adv. Math.* **48** (1983), 117–143.

May 2, 1994